8th International Congress of Croatian Society of Mechanics
29 September – 2 October 2015
Opatija, Croatia

Rocking motion of a single rigid rectangular block – analysis of the block slenderness assumption

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Abstract. Rocking motion of a single rigid rectangular block on rigid base subjected to base acceleration function, with the assumption that friction between block and base is large enough to prevent sliding motion, is analyzed. In literature, such analysis has been carried out using a linearized equation of motion assuming slender geometry of the block and small rotations. The analysis herein additionally addresses the cases when the block is non-slender and can undergo large rotations. With the aim to investigate the influence of assumptions introduced into linearized analysis on the accuracy of solution, three types of analysis are carried out: (1) linearized analysis assuming slender geometry of the block, (2) quasi-nonlinear analysis, i.e. linearized analysis taking real geometry of the block into account, and (3) nonlinear analysis. Newmark′s method with average acceleration is used for numerical integration and Newton-Raphson iterative procedure is used to solve the nonlinear equation. Free and forced rocking motion of slender and non-slender rigid rectangular block with varying initial conditions are analyzed. Analytical solutions of the linearized and quasi-nonlinear equation of motion are compared to numerical results.

1 Introduction

There are a number of structures that can be observed as discontinuous assemblies of bodies (blocks) when subjected to dynamic excitation of the ground. Such structures can be inherently discontinuous, either as a matter of convenience (e.g. ease of construction in structural masonry or dry stone walling) or as a deliberate strategy to avoid extensive thermal stresses (e.g. graphite cores in Advanced Gas Cooled Reactors, AGR, in nuclear power plants). Often these structures are deliberately discontinuous, organised as stacked and/or interlocked assemblies with a regular pattern and technologically intended gaps and clearances.

2 Analytical model

2.1 Equation of motion of a single block

Dynamic behavior of a single rigid rectangular block standing on a rigid base, with the assumption that friction between the block and base is high enough to prevent sliding, was first described by Housner [2]. If sliding motion is prevented, rigid block subjected to dynamic base excitation will either move translationally with the base or it will rock.

A simple dynamic model of a rocking rigid rectangular block on a rigid base subjected to horizontal base acceleration function is shown in Figure 1 .

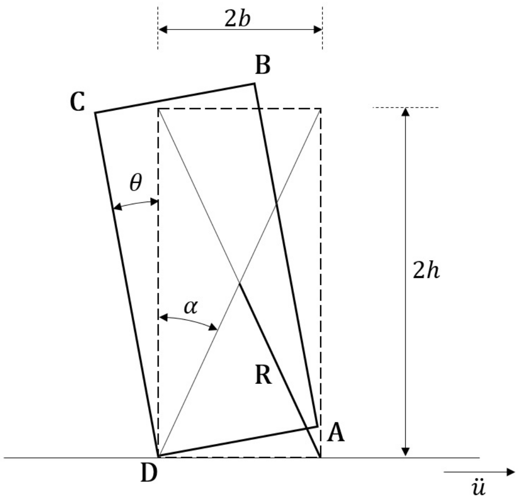

Figure 1: Rocking rigid block model

For a relatively small amplitude of the base acceleration function, the block will move along with the base without rocking. The equation of motion describing such behavior is

$$\Sigma M_D : mgR\sin\alpha - Yx = maR\cos\alpha, \tag{1}$$

where Y is the vertical reaction from the base and x is the distance between the point of application of this reaction and edge D. In the limiting case, when the block is about to start rocking, $x \to 0$ and

$$a = g\tan\alpha. \tag{2}$$

When the amplitude of the base acceleration function becomes larger than the right hand side of equation (2), rocking motion around corner A ($\theta < 0$), described with

$$I_0\ddot{\theta} = -mgR\sin(-\alpha - \theta) + mR\ddot{u}\cos(-\alpha - \theta), \tag{3}$$

or around corner D ($\theta > 0$), described with

$$I_0\ddot{\theta} = -mgR\sin(\alpha - \theta) + mR\ddot{u}\cos(\alpha - \theta), \tag{4}$$

is initiated. In the above equations I_0 is the moment of inertia with respect to one of the base corners, m is the mass of the block (see Figure 1), g is the gravity acceleration, R is the half-diagonal, α is the angle of slenderness of the block, θ and $\ddot{\theta}$ are the angular displacement (rotation) and angular acceleration of the block, respectively, and $\ddot{u}$ is the horizontal base acceleration function.

In order to use one equation to describe rocking motion in both directions, the following equation is derived [1]:

$$I_0\ddot{\theta} = -mgR\sin\left[\mathrm{sgn}\left(\theta\right)\alpha - \theta\right] + mR\ddot{u}\cos\left[\mathrm{sgn}\left(\theta\right)\alpha - \theta\right], \tag{5}$$

where $\mathrm{sgn}\left(\theta\right)$ is the standard sign function.
The horizontal base acceleration function is here taken as harmonic

$$\ddot{u} = a_0\sin\left(\omega t\right), \tag{6}$$

where a_0 is its amplitude, and ω its circular frequency, while t is the time.
When the block switches from rocking around corner A to rocking around corner D (or the other way around) it experiences an impact with the base. In this work it is assumed that there is no loss of energy due to this impact.

2.2 Linearization of the equation of motion

In literature, a linearized version of equation (5) is usually found [2, 1]. However, it is important to distinguish two separate steps in the procedure of linearizing basic equation of motion. The sine and cosine functions in equation (5), can be substituted with the following if the block experiences small rotations θ:

$$\sin\left[\mathrm{sgn}\left(\theta\right)\alpha - \theta\right] = \mathrm{sgn}\left(\theta\right)\sin\left(\alpha\right) - \theta\cos\left(\alpha\right) \tag{7}$$

and

$$\cos\left[\mathrm{sgn}\left(\theta\right)\alpha - \theta\right] = \cos\left(\alpha\right) + \mathrm{sgn}\left(\theta\right)\theta\sin\left(\alpha\right). \tag{8}$$

If only positive direction of excitation and positive rotations are observed, equation (5) becomes

$$I_0\ddot{\theta} = mR\theta\left[a_0\sin\left(\omega t\right)\sin\left(\alpha\right) + g\cos\left(\alpha\right)\right] + mR\left[a_0\sin\left(\omega t\right)\cos\left(\alpha\right) - g\sin\left(\alpha\right)\right]. \tag{9}$$

No limits are put on the geometry of the block in the above equation, but it is still linear with respect to the unknown rotation θ. This equation will be referred to as *quasi-nonlinear* equation of motion of a single rigid rectangular block.

Assuming that the block is slender (the angle of slenderness, α, is small) and the rotations are also small, equation (9) further simplifies to the following form:

$$\ddot{\theta} - p^2\theta = -p^2\left[\alpha - \frac{a_0}{g}\sin\left(\omega t\right)\right], \tag{10}$$

with $p = \sqrt{\frac{mgR}{I_0}}$ as the so-called frequency parameter, which is a second-order non-homogeneous differential equation with constant coefficients. Analytical solution of this differential equation in case of free rocking motion ($\omega = 0$) with initial conditions at time $t = 0$ given as $\theta\left(0\right) = \theta_0$ and $\dot{\theta}\left(0\right) = 0$ is

$$\theta\left(t\right) = \left(\theta_0 - \alpha\right)\cosh\left(pt\right) + \alpha. \tag{11}$$

Analytical solution of differential equation (10) in case of forced rocking motion given with eq. (6) with initial conditions at time $t = 0$ given as $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ is:

$$\theta(t) = -\alpha \cosh(pt) + \frac{\omega p \frac{a_0}{g}}{\omega^2 + p^2} \sinh(pt) + \alpha - \frac{p^2 \frac{a_0}{g}}{\omega^2 + p^2} \sin(\omega t). \tag{12}$$

The analytical solution enables defining an analytical condition for overturning of the block due to given base acceleration function during motion with positive rotation. Starting with the instant when the block rotates back to vertical position and its bottom side hits the base, the above equation and its analytical solution do not longer apply. A differential equation such as equation (10), but with signs corresponding to negative rotation and initial conditions written at time of impact with the base, can be solved in order to obtain analytical solution in that case.

In an attempt to derive an analytical solution of the equation of motion not limited to slender blocks but only to small rotations, i.e. of the quasi-nonlinear equation of motion, equation (9) becomes

$$\ddot{\theta} - \frac{mR}{I_0}\left[g\cos(\alpha) + a_0 \sin(\omega t)\sin(\alpha)\right]\theta = -p^2\left[\sin(\alpha) - \frac{a_0}{g}\cos(\alpha)\sin(\omega t)\right], \tag{13}$$

which is a second-order non-homogeneous differential equation with non-constant coefficients.

2.3 *Time integration*

Equation of motion (10) written at a time step i (for any sign of rotation) is:

$$I_0\ddot{\theta}_i - mgR\theta_i = -\operatorname{sgn}(\theta_i)\, mgR\alpha + mRa_0 \sin(\omega t_i). \tag{14}$$

Using Newmark's integration method [3], angular acceleration can be written as:

$$\ddot{\theta}_i = \frac{\theta_i - \theta_{i-1} - \Delta t\dot{\theta}_{i-1} - \left(\frac{1}{2} - \beta\right)\Delta t^2 \ddot{\theta}_{i-1}}{\beta \Delta t^2}, \tag{15}$$

and substituted into eq. (14), which gives the following equation:

$$I_0 \frac{\theta_i - \theta_{i-1} - \Delta t\dot{\theta}_{i-1} - \left(\frac{1}{2} - \beta\right)\Delta t^2 \ddot{\theta}_{i-1}}{\beta \Delta t^2} - mgR\theta_i = -\operatorname{sgn}(\theta_i)\, mgR\alpha + mR\ddot{u}_i. \tag{16}$$

In the above equation θ_{i-1}, $\dot{\theta}_{i-1}$ and $\ddot{\theta}_{i-1}$, the rotation, angular velocity and angular acceleration of the block at time step $i-1$ are known, and θ_i, rotation of the block at time step i, is the only unknown. It is important to note that the unknown θ_i also appears inside the sign function. Therefore, when using Newmark's method of integration and solving for unknowns θ_i, $\dot{\theta}_i$ and $\ddot{\theta}_i$ at time step i, it is necessary to use a procedure that will take that into account. For that purpose, two solutions for the unknown rotation, θ_i, are computed at each time step: a solution with the assumption that $\theta_i \leq 0$ and a solution with the assumption that $\theta_i > 0$. The correct solution is selected by checking whether the sign of θ_i obtained from equation (16) corresponds to the assumed sign of θ_i.

2.4 Nonlinear equation of motion and solution procedure

When the assumption of small unknown rotations is not justifiable, nonlinear equation (5) needs to be solved. In order to find a solution at each time step using Newmark's method of integration, equation (15) can be substituted into equation (5) to obtain equation:

$$F\left(\theta_i\right) \equiv \frac{I_0}{\beta \Delta t^2}\left[\theta_i - \theta_{i-1} - \Delta t \dot{\theta}_{i-1} - \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{\theta}_{i-1}\right] + mgR \sin\left[\operatorname{sgn}\left(\theta_i\right)\alpha - \theta_i\right] - mR\ddot{u}_i \cos\left[\operatorname{sgn}\left(\theta_i\right)\alpha - \theta_i\right] = 0. \quad (17)$$

To solve the nonlinear equation, Newton-Raphson iterative procedure is used.

Inside each iteration, j, the function $F\left(\theta_{i,j}\right)$ given in equation (17) needs to be calculated. If the value of this function is lower than a given tolerance, solution from the last iteration, $\theta_{i,j}$, is the value of the rotation at that time step, θ_i. Otherwise, the first derivative of the function $F\left(\theta_{i,j}\right)$ with respect to the unknown $\theta_{i,j}$ needs to be calculated as

$$F'\left(\theta_{i,j}\right) = \begin{cases} \frac{I_0}{\beta \Delta t^2} + \left(\pm\alpha\infty - 1\right) mgR \cos\left[\operatorname{sgn}\left(\theta_{i,j}\right)\alpha - \theta_{i,j}\right] \\ \quad + \left(\pm\alpha\infty - 1\right) mR\ddot{u}_i \sin\left[\operatorname{sgn}\left(\theta_{i,j}\right)\alpha - \theta_{i,j}\right] & \text{if } \theta_{i,j} = 0 \\ \frac{I_0}{\beta \Delta t^2} - mgR \cos\left[\operatorname{sgn}\left(\theta_{i,j}\right)\alpha - \theta_{i,j}\right] \\ \quad - mR\ddot{u}_i \sin\left[\operatorname{sgn}\left(\theta_{i,j}\right)\alpha - \theta_{i,j}\right] & \text{if } \theta_{i,j} \neq 0. \end{cases} \quad (18)$$

The above equation shows that the derivative $F'\left(\theta_{i,j}\right)$ is infinitely large when $\theta_{i,j} = 0$ because the derivative of the signum function is the Dirac delta function. It is possible to calculate the exact value of this derivative for $\theta_{i,j} \in (-\infty, 0) \cup (0, +\infty)$, while the derivative for $\theta_{i,j} \equiv 0$ would make Newton-Raphson iterative procedure to crash. For this reason the derivative $F'\left(\theta_{i,j}\right)$ is assumed to be

$$F'\left(\theta_{i,j}\right) = -mgR \cos\left[\operatorname{sgn}\left(\theta_{i,j}\right)\alpha - \theta_{i,j}\right] - mR\ddot{u}_i \sin\left[\operatorname{sgn}\left(\theta_{i,j}\right)\alpha - \theta_{i,j}\right] \quad (19)$$

for every value of $\theta_{i,j}$.

3 Numerical results

In order to compute the solution of rotation, angular velocity and angular acceleration of single rigid rectangular block on rigid base subjected to horizontal base acceleration function at each time step, three *MatLAB* codes were developed: the first one based on linearized equation of motion, the second one based on quasi-nonlinear equation of motion and the third one based on nonlinear equation of motion.

3.1 Slender block

The analyses were carried out for a slender rigid block with width 11,5 cm, height 50 cm and mass 6.329 kg (Figure 2). Angle of slenderness of the block is $\alpha = 0.2261\ rad$, half-diagonal is $R = 0.2565\ m$ and frequency parameter is $p = 5.3558\ \frac{rad}{s}$.

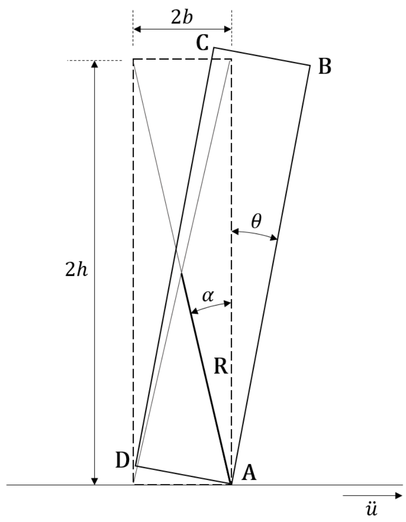

Figure 2: Slender rigid rectangular block analyzed

Analysis of free rocking motion of the given slender block with no energy loss ($\eta = 1$) and with initial rotation $\theta_0 = 0.15\ rad$ is carried out with time step size $dt = 0.0005\ s$ for 5 s. Figure 3 shows results obtained from linearized (blue line) and nonlinear (red dashed line) analysis.

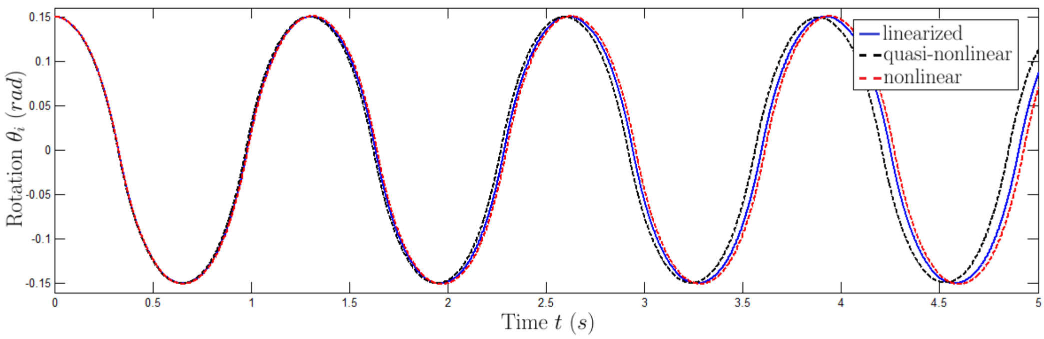

Figure 3: Rotation time-history for a free rocking motion of a rigid rectangular block with no energy loss (linearized and nonlinear analysis)

Period of motion in nonlinear analysis is slightly larger than in the linearized analysis, but both values are comparable to results of the analytical solution of the equation of motion given by Housner [2] (Table 1):

$$T_{Housner} = \frac{4}{p}\cosh^{-1}\left(\frac{1}{1-\frac{\theta_0}{\alpha}}\right). \tag{20}$$

Calculation method	Marking	Value
Housner's approximate solution [2]	$T_{Housner}$	$1.309\ s$
Linearized numerical analysis	T_{lin}	$1.307\ s$
Nonlinear numerical analysis	T_{nonlin}	$1.314\ s$
Quasi-nonlinear numerical analysis	$T_{q-nonlin}$	$1.293\ s$

Table 1: Rocking period using approximate analytical solution [2], linearized, quasi-nonlinear and nonlinear numerical analysis for a slender rigid block

Two series of analyses of forced rocking motion with no energy loss are carried out. The forcing function given with eq. (6) with amplitudes between $0.1\alpha g\ \frac{m}{s^2}$ and $5\alpha g\ \frac{m}{s^2}$ and frequencies between $\pi\ \frac{rad}{s}$ and $20\pi\ \frac{rad}{s}$ is used. All the analyses last 10 s and the final outcomes of the analyses are shown as results and compared. Figure 4 shows results obtained by linearized analysis, Figure 5 shows results obtained by quasi-nonlinear analysis, while Figure 6 shows results obtained by nonlinear analysis. Red rectangles mark the parameters of ground excitation which led to overturning of the block during the analysis, while blue rectangles mark the parameters that led to rocking behaviour but no overturning of the block.

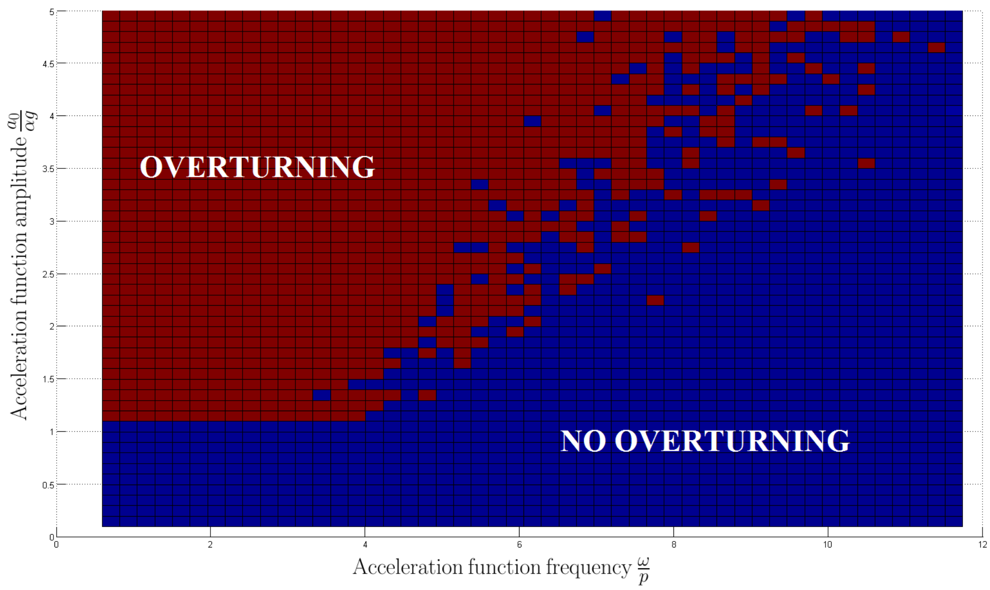

Figure 4: Overturning/non-overturning area depending on frequency and amplitude of ground acceleration function obtained during 10 seconds of linearized analysis of rocking motion of slender rigid rectangular block

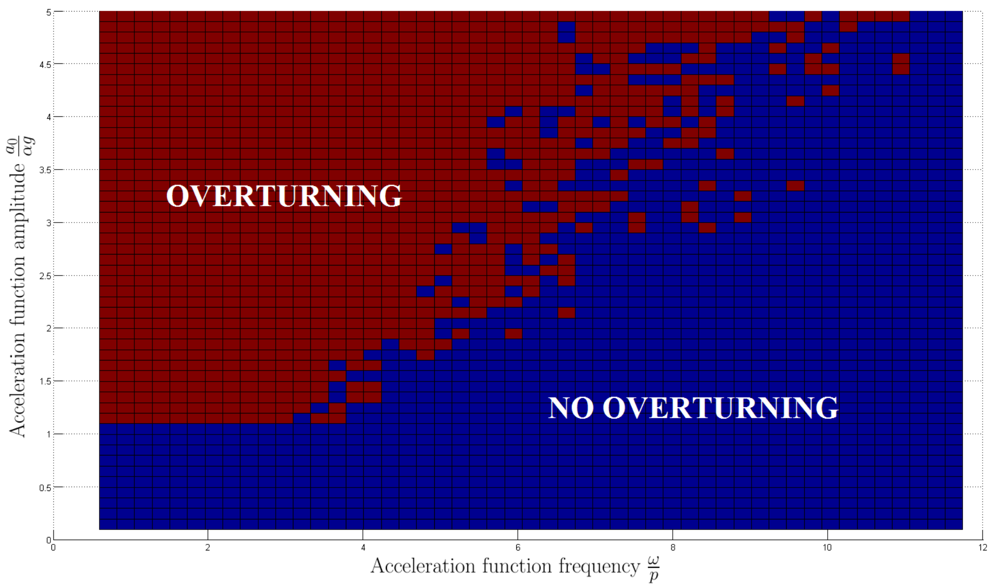

Figure 5: Overturning/non-overturning area depending on frequency and amplitude of ground acceleration function obtained during 10 seconds of quasi-nonlinear analysis of rocking motion of slender rigid rectangular block

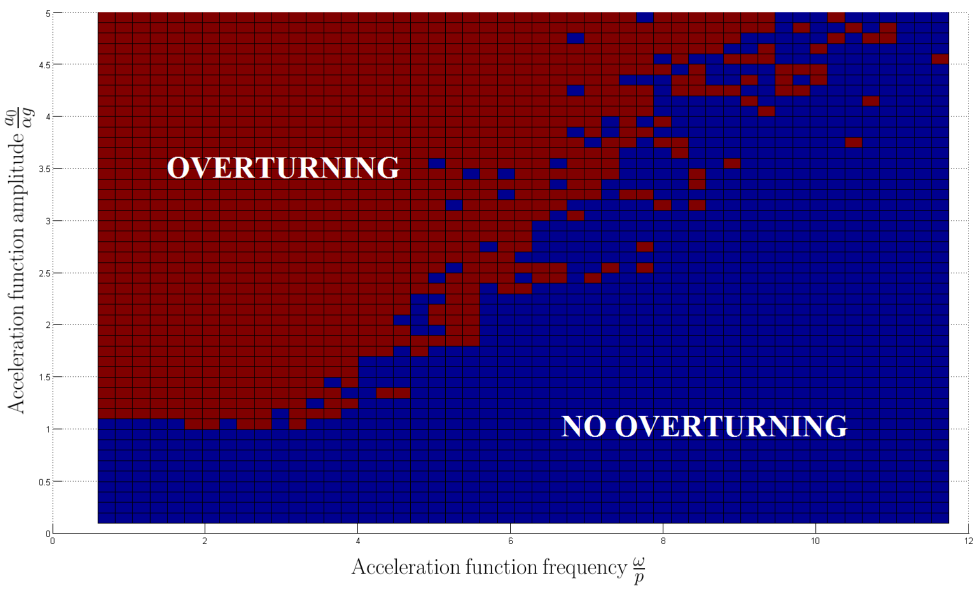

Figure 6: Overturning/non-overturning area depending on frequency and amplitude of ground acceleration function obtained during 10 seconds of nonlinear analysis of rocking motion of slender rigid rectangular block

Linearized, quasi-nonlinear and nonlinear numerical analysis of dynamic behaviour of slender rigid rectangular block under harmonic horizontal ground excitation result in similar areas of overturning. This result is expected, since rocking (no overturning) is only possible for small rotations, for which the difference between a linear and a nonlinear analysis becomes negligible. Also, for a slender block $\sin(\alpha) \approx \alpha$ and $\cos(\alpha) \approx 1$ and the difference between the linearized and the quasi-nonlinear analysis also becomes negligible.

3.2 Nonslender block

All the analyses carried out in the previous section were dealing with rigid rectangular blocks that can be considered slender because of having height to width ratio lower than 2.8. Blocks with height to width ratio greater than 2.8 [2] have large angle of slenderness, therefore cannot be considered slender, and can rock with large rotations.

Analyses are now carried out for a non-slender rigid block with width 50 cm, height 50 cm and mass 27.5 kg (Figure 7). Angle of slenderness of the block is $\alpha = 0.7854\ rad$, half-diagonal is $R = 0.3536\ m$ and frequency parameter is $p = 4.5618\ \frac{rad}{s}$.

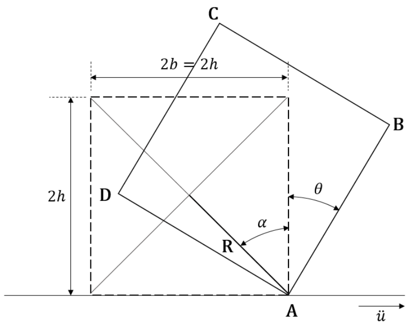

Figure 7: Model of analyzed non-slender rigid rectangular block

Results of the linearized and the nonlinear numerical analysis of rocking motion of a non-slender rigid rectangular block are shown in Figure 8. Initial rotation of the block is $\theta_0 = 0.75\ rad$. As expected, the results of the nonlinear numerical analysis (red dashed line in Figure 8) now differ from the results of the linearized analysis more than before (blue solid line in Figure 8). The period of rocking motion is somewhat shorter in the nonlinear numerical analysis (Figure 8 and Table 2), while in the quasi-nonlinear analysis it is significantly shorter.

Two series of analyses of forced rocking motion with no energy loss are carried out. The forcing function given with eq. (6) with amplitudes between $0.1\alpha g\ \frac{m}{s^2}$ and $5\alpha g\ \frac{m}{s^2}$ and frequencies between $\pi\ \frac{rad}{s}$ and $20\pi\ \frac{rad}{s}$ is used. All the analyses last 10 s and the final outcomes of the analyses are shown as results and compared. Figure 9 shows results obtained from linearized analysis, Figure 10 shows results from the quasi-nonlinear analysis, while Figure 11 shows results obtained from nonlinear analysis. Red rectangles mark the parameters of ground excitation which led to overturning of the block during the analysis,

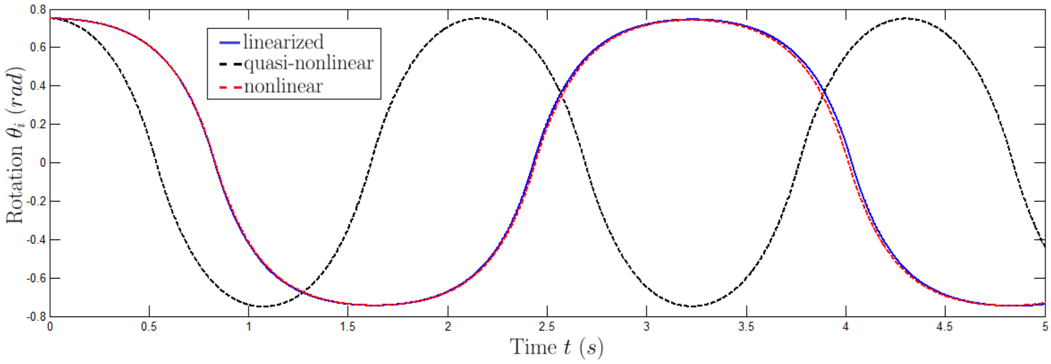

Figure 8: Rotation time-histories of free rocking motion of non-slender rigid rectangular block with no energy loss (linearized and nonlinear analysis)

Calculation method	**Marking**	**Value**
Housner's approximate solution [2]	$T_{Housner}$	3.325 s
Linearized numerical analysis	T_{lin}	3.236 s
Nonlinear numerical analysis	T_{nonlin}	3.17 s
Quasi-nonlinear numerical analysis	$T_{q-nonlin}$	2.15 s

Table 2: Period of free rocking using approximate numerical solution [2] and linearized, quasi-nonlinear and nonlinear numerical analysis for non-slender rigid block

while blue rectangles mark the parameters that led to rocking behaviour but no overturning of the block.

Linearized and nonlinear numerical analysis result in completely different areas of overturning. The horizontal boundary between the overturning and non-overturning area visible in the linearized analysis (Figure 9) is also visible in the nonlinear analysis, but corresponds to larger amplitudes of the acceleration function (Figure 11), as it also does for the quasi-nonlinear analysis. Note that in the first case the condition for initiation of rocking is $a > g\alpha$, while in the other two cases the condition is $a > g\tan(\alpha)$, which explains this result. In this example the linearized analysis grossly underestimates the stability of the non-slender rocking block. Quasi-nonlinear analysis (Figure 10) shows area of overturning similar to the nonlinear analysis, although it slightly overestimates stability of the block. Even though one may argue that the quasi-nonlinear analysis could be used for non-slender blocks, these results show that it gives non-conservative estimate for the condition of overturning/no overturning of the block. There are cases of rocking motion in quasi-nonlinear analysis where rotations are large and the assumption of small rotations is not justifiable. In these cases the linearized analysis and the quasi-nonlinear analysis both give inaccurate results and nonlinear analysis has to be carried out.

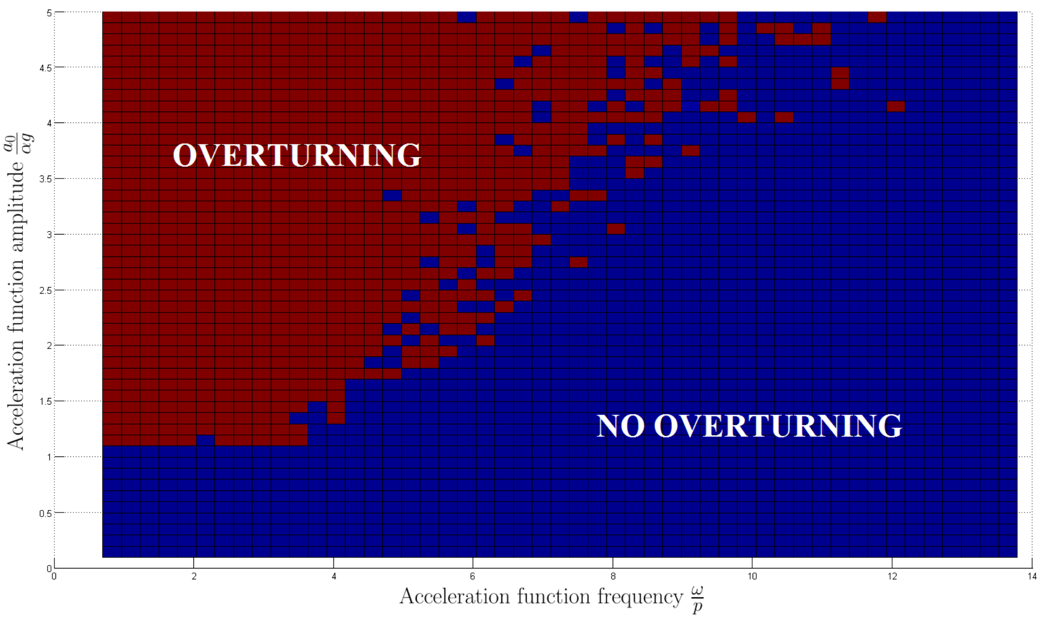

Figure 9: Overturning/non-overturning area depending on frequency and amplitude of ground acceleration function obtained during 10 seconds of linearized analysis of rocking motion of non-slender rigid rectangular block

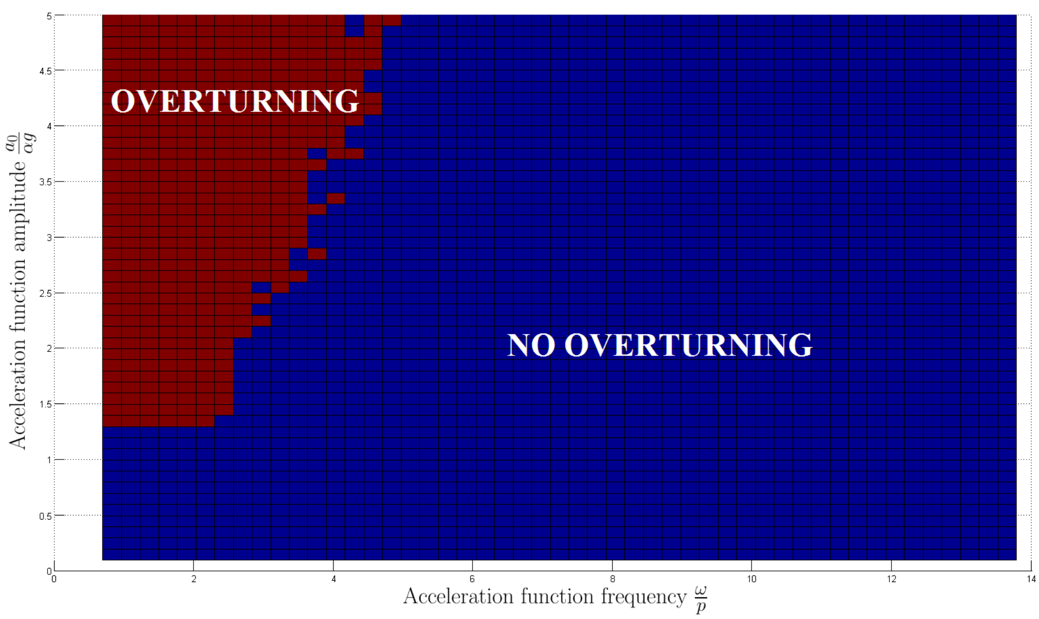

Figure 10: Overturning/non-overturning area depending on frequency and amplitude of ground acceleration function obtained during 10 seconds of quasi-nonlinear analysis of rocking motion of non-slender rigid rectangular block

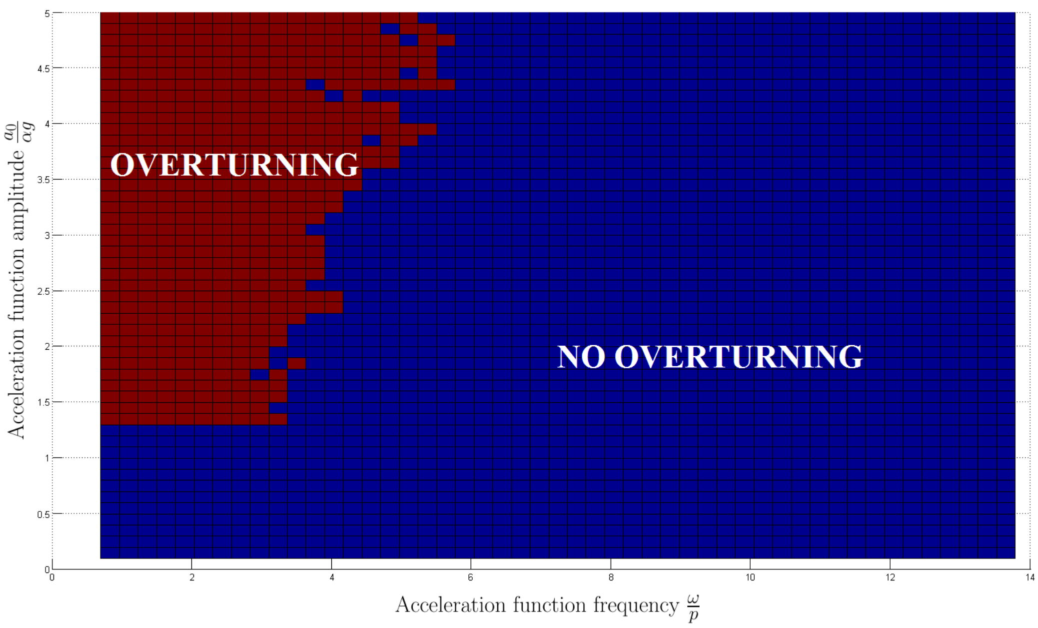

Figure 11: Overturning/non-overturning area depending on frequency and amplitude of ground acceleration function obtained during 10 seconds of nonlinear analysis of rocking motion of non-slender rigid rectangular block

4 Conclusions and future work

Equation of motion of a single rigid rectangular block on a rigid base subjected to horizontal base acceleration function, with the assumption that friction between block and base is high enough to prevent sliding, is numerically solved. Depending on the assumptions that either the block is slender and experiences small rotations or it just experiences small rotations (regardless on the geometry of the block) there are two versions of equation of motion: linearized and quasi-nonlinear equation of motion, in addition to the original nonlinear equation.

Linearized and nonlinear equation of motion are solved numerically at discrete time steps using *Matlab* and compared for both slender and non-slender rigid rectangular block. Free rocking motion and forced rocking motion are observed. Analyses showe that, while the linearized and the nonlinear procedure result in similar dynamic behavior for slender blocks, dynamic behavior and outcomes (such as overturning) obtained from these analyses significantly differ for non-slender blocks. Quasi-nonlinear and nonlinear procedure result in similar behaviour for both slender and non-slender blocks but in the case of non-slender block the quasi-nonlinear analysis slightly overestimates block's stability and thus the nonlinear analysis should be carried out to estimate block's stability, regardless of the numerical cost.

Acknowledgement

These results were obtained within the research project 3/13 Evidence based characterisation of dynamic sensitivity for multiblock structures computational simulation and experimental validation financially supported by the Unity Through Knowledge foundation.

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